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Six Modes of Triply Super Stable Kneading Sequences in 1D Trimodal Maps

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Abstract

Triply super stable kneading sequences (TSSKS) are very important kernel concept in the study of symbolic dynamics of 1D trimodal maps. For a given period *n*, there are six types of TSSKS in which two of them decide the six cyclic star products, others supplemented the 'joints' in the symbolic space. For the former, start products provide the method to research metric universalities in the period-*n*-tupling process, the devil's staircase of topological entropy and self-similar bifurcation structure in classical dynamical systems, the later will be calculated and obtained the corresponding parameters which can occupy so called the admissibe region. In this paper, firstly, for a period *n* takes 3-12, we produce all the permutations of six types $(ZEXDYC)^\infty$, $(XDZEYC)^\infty$, $\left((ZE)^\infty,(XD)^\infty,(YC)^\infty\right)$, $\left((ZEXD)^\infty,(YC)^\infty\right)$, $\left((ZEYC)^\infty,(XD)^\infty\right)$ and $\left((XDYC)^\infty,(ZE)^\infty\right)$, by the famous admissibilty conditions, the admissible sets κ_n^m are obtained respectively, here *m* stands for the mode of the TSSKS and takes integer 0-5; second, for $W \in \kappa_n^m$, W detemines a system of nonlinear equations uniquely by passing three critical points C, D and E, *m* ensures six different modes of equations, for an proper initial value, the newton-iteration method is applied to get the three parameters of W. For *m* takes 2-5, these parameters of TSSKS in κ_n^m are calculated firstly in the paper, it would describe the parameter space and boundaries and enhence the knowledge of symbolic dynamics of 1D trimodal maps.

Keywords

Symbolic Dynamics, TSSKS, Admissibility Conditions, Parameter Calculation.

1. Introduction

Symbolic dynamics is an important tool to probe the classic dynamic system [1]. Dual star products and cyclic star products have played an important role in research the bifurcation process with self-similar fractal structure and Feigenbaum' constants are generalized in 1D bimodal and trimodal maps [2-3]. In fact, we have realized super-stable kneading sequence (SSKS) is key substance to develop the corresponding symbolic dynamics, for example, TSSKS is SSKS in trimodal maps, $(ZEXDYC)^\infty$ and $(XDZEYC)^\infty$ are TSSKS of mode 0 and 1 with single cycle which determine the six cyclic star products, while they are called 'joint' while other admissible sequence who do not pass all the critical points form 'skeleton' and 'muscle' by Mackay and Tresser [4], they have given the construction of symbolic space. Kaplan [5] had proposed an algorithm to solve the inverse problem that for a SSKS the corresponding parameter could be calculated firstly, it was developed by B.-L. Hao in symbolic dynamics of unimodal maps and named it the word-lifting algorithm [6]. With the development of the 1D symbolic dynamics, the difficult of lacking explicit express in inverse function of polynomials to the 5th power or more leads to no way to research the symbolic dynamics in 1D *m*-modal maps if $m \geq 4$. We had proposed an effective numerical algorithm and solved this problem [7], in fact, the new word-lifting algorithm is quick and effective because of using Newton's iterative method of second-order convergence, the key issue is a proper initial point should be given in the parameter space, else the iterative process may lead to a divergence. By the new word-lifting algorithm, the structure of symbolic space may be generalized to the parameter space. In fact, there are other four modes TSSKS in 1D trimodal maps, mode 2 is $\left((ZE)^{\infty},(XD)^{\infty},(YC)^{\infty}\right)$ with three single cycle, modes 3-5 are $\left((ZEXD)^{\infty},(YC)^{\infty}\right)$, $\left((ZEYC)^{\infty},(XD)^{\infty}\right)$ and $\left((XDYC)^{\infty},(ZE)^{\infty}\right)$ with two cycles and single cycle respectively. Another definition of SSKS is that period sequences corresponding parameters could be obtained by the word-lifting algorithm. All these TSSKS with six modes or types are worthy of researching furtherly. In the paper, for a given period *n*, we produce all the permutation of four letters *L*. *M*, *N* and *R*, insert three critical points *C*, *D* and *E* in turn. By the order rule of sequences and the admissibility conditions, TSSKS period *n* from 3 to 12 are store in sets κ_n^m, *m* = 0, 1,2,3,4,5. On the other hand, we designed six types of equations as by six modes of the TSSKS, the corresponding parameters are calculated, the orbit iterative graphs are drawn for examples.

2. The Method of Producing Admissible Sets κ_n^m of Trimodal Maps

2.1. Symbolic Dynamics of 1D Trimodal Maps

Consider the general trimodal maps $f(x)=k\int(x-c)(x-d)(x-e)dx+b$, on the interval [-1,1] of endomorphism, by two boundary conditions *f*(-1) = -1 and *f*(1) = -1, $k=\dfrac{-4(1+b)}{1+2(c*d+c*e+d*e)}$, $d=\dfrac{-(c+e)}{1+3*c*e)}$, parameters *k* and *d* are eliminated . In fact, the iterated map of trimodal is written as:

$$x_{n+1}=f(x_n,c,e,b) \tag{1}$$

Here, *c*, *d* and *e* are horizontal coordinates of three critical points *C, D* and *E*, while *L*, *M, N* and *R* are four monotonous limbs, by the MSS order [8], $L \prec C \prec M \prec D \prec N \prec R$ holds. For an initial point x_0, by iterative map (1), a numerical orbit is obtained as x_0, x_1, ... , x_n, it can be converted into a symbolic sequence $S_0 S_1 \cdots S_n \cdots$, by the following rule (2), the coarse granulation process is quintessence of the symbolic dynamics [9].

$$S_k=\begin{cases} L, & if\ \ -1<x_k<c, \\ C, & if\ \ x_k=c, \\ M, & if\ \ c<x_k<d, \\ D, & if\ \ x_k=d, \qquad k\in Z^{+} \\ N, & if\ \ d<x_k<e, \\ E, & if\ \ x_k=e, \\ R, & if\ \ e<x_k<1. \end{cases} \tag{2}$$

If the sequence is periodic and passing through the three critical points, they are called TSSKS. For example, $(ZEXDYC)^{\infty}$, $(XDZEYC)^{\infty}$, $\left((ZE)^{\infty},(XD)^{\infty},(YC)^{\infty}\right)$, $\left((ZEXD)^{\infty},(YC)^{\infty}\right)$, $\left((ZEYC)^{\infty},(XD)^{\infty}\right)$ and $\left((XDYC)^{\infty},(ZE)^{\infty}\right)$, they are marked as mode 0,1,2,3,4 and 5 in turn respectively. For conciseness, they are normalized to the simple forms $ZEXDYC$, $XDZEYC$, (ZE,XD,YC),$(ZEXD,YC)$,$(ZEYC,XD)$ and $(XDYC,ZE)$. The mode 0 and 1 are single cycle TSSKS which determine the six cyclic star product laws; the mode 2 is with three single cycles, other modes are with one two cycles and one single cycle, where X,Y and Z are sequences composed of $\{L,M,N,R\}$. The six types of TSSKS are called 'joint' in the symbolic space, however, we hope to generalize them to parameter space for the furtherly research of symbolic dynamics. (See Table 1)

Table 1. Six modes of TSSKS in 1D trimodal maps

TSSKS	$ZEXDYC$	$XDZEYC$	(ZE,XD,YC)	$(ZEXD,YC)$	$(ZEYC,XD)$	$(XDYC,ZE)$
Mode	0	1	2	3	4	5

2.2. The Order Rule of Sequences and Admissibility Conditions

Here, we know M and R are monotonous descending limbs while L and N are monotonous ascending limbs. The count of M and R stands for the parity of compound functions (1), so in order to compare two sequences by (2), if sequence W is composed of $\{L,M,N,R\}$ the operator τ is defined as:

$$\tau(W)=\begin{cases}1, & if\ \text{the count of M and R is even;}\\ -1, & if\ \text{the count of M and R is odd.}\end{cases} \tag{3}$$

2.2.1. The Order Rule of Sequences

The order rule of sequences comes from the parity of compound functions and the symbolic MSS order $L \prec C \prec M \prec D \prec N \prec R$. For two sequences $W_1 = \triangle w_1 \cdots$ and $W_2 = \triangle w_2 \cdots$, here $w_1, w_2 \in \{L,C,M,D,N,E,R\}$ and $\triangle$ is the common leading string composed of letters from $\{L,M,N,R\}$. If $\tau(\triangle)=1$ and $w_1 \prec w_2$, then $W_1 \prec W_2$, if $\tau(\triangle)=1$ and $w_2 \prec w_1$,then $W_2 \prec W_1$; If $\tau(\triangle)=-1$ and $w_1 \prec w_2$, then $W_2 \prec W_1$, if $\tau(\triangle)=-1$ and $w_2 \prec w_1$,then $W_1 \prec W_2$. For example, $W_1 = RLC$, $W_2 = LC$, the common leading string $\triangle = \phi$, $\tau(\triangle)=1$, $\because L \prec R$, $\therefore W_2 \prec W_1$.

2.2.2. Admissibility Conditions

B.-L. Hao used a shift operator $\sigma^k(W)$ to express all subsequences of every letter in W in the famous admissibility condition in 1D unimodal maps. Here, we introduce a notation $\overline{S}(W)$ stands for all the subsequences of W in the admissibility conditions (4) of 1D trimodal maps, the notation is inherent and concise.

$$\begin{cases}\overline{L}(W) \prec \overline{C}(W), \overline{M}(W) \prec \overline{C}(W);\\ \overline{D}(W) \prec \overline{M}(W), \overline{D}(W) \prec \overline{N}(W);\\ \overline{N}(W) \prec \overline{E}(W), \overline{R}(W) \prec \overline{E}(W).\end{cases} \tag{4}$$

$\overline{C}(W)$ stands for the height of peak C in symbolic space, so it is higher than height of the left limb L $\overline{L}(W)$ and height of the right limb M $\overline{M}(W)$; $\overline{E}(W)$ stands for the height of peak E in symbolic space, so it is higher than height of the left limb N $\overline{N}(W)$ and height of the right limb R $\overline{R}(W)$; $\overline{D}(W)$ stands for the height of valley D in symbolic space, so it is lower than height of the left limb M $\overline{M}(W)$ and height of the right limb N $\overline{N}(W)$.

However, the TSSKS has six mode 0-5, $\overline{C}(W)$, $\overline{D}(W)$ and $\overline{E}(W)$ have different expressions for every mode of TSSKS, we present the key part (5) for the convenience of programming by (4) for all modes of TSSKS.

$$\begin{cases} \overline{C}(W)=ZE, & \overline{D}(W)=YC, & \overline{E}(W)=XD, & \text{if mode}=0; \\ \overline{C}(W)=XD, & \overline{D}(W)=ZE, & \overline{E}(W)=YC, & \text{if mode}=1; \\ \overline{C}(W)=YC & \overline{D}(W)=XD, & \overline{E}(W)=ZE, & \text{if mode}=2; \\ \overline{C}(W)=YC, & \overline{D}(W)=ZE, & \overline{E}(W)=XD, & \text{if mode}=3; \\ \overline{C}(W)=ZE, & \overline{D}(W)=XD, & \overline{E}(W)=YC, & \text{if mode}=4; \\ \overline{C}(W)=XD, & \overline{D}(W)=YC, & \overline{E}(W)=ZE, & \text{if mode}=5. \end{cases} \tag{5}$$

2.3. Constructing Admissible Sets κ_n^m With Period-*n* for All Six Modes of TSSKS

Here, for a given period *n*, the goal of this part is to get the admissible sets κ_n^m for *m* = 0,1,2,3,4,5. First, there are n-3 seats for four letters from $\{L,M,N,R\}$ to sit, the permutation number is 4^{n-3}, one of the method is to transform decimal integer $0\sim 4^{n-3}$ to quaternary integer, then replace 0,1,2,3 with '*L*', '*M*','*N*' and '*R*' respectively, another simpler method is to use Python language,

from itertools import product
LMNR = [''.join(k) for k in product('LMNR',repeat = n-3)]

Table 2. The counts of admissible sets κ_n^m with period *n* from 3 to 13

TSSKS / Period *n*	*m* = 0	*m* = 1	*m* = 2	*m* = 3	*m* = 4	*m* = 5	Total
	$ZEXDYC$	$XDZEYC$	(ZE,XD,YC)	$(ZEXD,YC)$	$(ZEYC,XD)$	$(XDYC,ZE)$	
3	1	0	0	0	0	1	2
4	4	0	1	0	2	4	11
5	17	1	5	3	9	13	48
6	61	9	24	15	37	45	191
7	214	47	100	66	141	157	725
8	746	214	394	272	516	560	2702
9	2598	916	1505	1089	1869	2014	9991
10	9074	3778	5698	4285	6767	7298	36900
11	31922	15229	21464	16717	24571	26616	136519
12	113212	60528	80707	64887	89702	97555	506591
13	404837	238393	303763	250946	329571	359342	1886852

The value of *n* should be given in advance, variable LMNR stores 4^{n-3} strings, then traversing these string, insert *C*, *D* and *E*, there are $(n-2)(n-1)4^{n-3}$ sequences in the innermost loop, then produce six tuples like TSSKS in Table 1, then applying function if_admissible to every tuple, if it satisfies (4) and (5), then store it to a pandas DataFrame object, finally save it to a xlsx or pickle file. Table 2 shows for period 3-13 and *m* 0-5, the counts of admissible TSSK, the data is obtained by a series of python programs which are so long to be omitted here.

The admissible set κ_{13}^{m} has been saved to a pickle file about 200Mb, κ_{14}^{m} would spend more CPU time and more storage disk space, so we do not consider the case $n \geq 14$. In fact, these TSSKS are so many to be experimental data for the word-lifting algorithm on describing the shape of parameter space. It provides important data for researching the symbolic dynamics of 1D trimodal maps.

3. Parameter Calculating for Six Types of TSSKS

3.1. Nonlinear Equations Determined By the Six type of TSSKS

Let $ZE = z_1 z_2 \cdots z_r E = \left(\bigcup_{i=1}^{r} z_i \right) E$, $XD = x_1 x_2 \cdots x_s D = \left(\bigcup_{i=1}^{s} x_i \right) D$ and $YC = y_1 y_2 \cdots y_t C = \left(\bigcup_{i=1}^{t} y_i \right) C$, $n = r + s + t + 3 = |ZE| + |XD| + |YC|$, $r, s, t \in Z^{+}$, $z_i, x_i, y_i \in \{L, M, N, R\}$, *n* is the period of TSSKS. It is easy to find in Table 1 that six modes of TSSKS are different combinations of ZE, XD and YC. The inherent of TSSKS is passing through three critical points, so there are three nonlinear equations determined by an arbitrary $W \in \kappa_n^m$, $f_L^{-1}, f_M^{-1}, f_N^{-1}$ and f_R^{-1} stand for the inverse limb function of L, M, N and R in (1). (6) and (7) are systems of nonlinear equations of TSSKS in κ_n^0 and κ_n^1 respectively. Mode 0 and Mode 1 stand for one single cycle TSSKS.

$$\begin{cases} F_1(c,e,b) = f(c,c,e,b) - f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots f_{z_r}^{-1}(e,c,e,b) = 0 \\ F_2(c,e,b) = f(e,c,e,b) - f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots f_{x_s}^{-1}(d,c,e,b) = 0 \\ F_3(c,e,b) = f(d,c,e,b) - f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots f_{y_t}^{-1}(c,c,e,b) = 0 \end{cases} \tag{6}$$

$$\begin{cases} F_1(c,e,b) = f(c,c,e,b) - f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots f_{z_r}^{-1}(e,c,e,b) = 0 \\ F_2(c,e,b) = f(e,c,e,b) - f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots f_{x_s}^{-1}(d,c,e,b) = 0 \\ F_3(c,e,b) = f(d,c,e,b) - f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots f_{y_t}^{-1}(c,c,e,b) = 0 \end{cases} \tag{7}$$

(8) is a system of nonlinear equations of TSSKS in κ_n^2. Mode 2 stands for three one single cycle TSSKS.

$$\begin{cases} F_1(c,e,b) = f(e,c,e,b) - f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots f_{z_r}^{-1}(e,c,e,b) = 0 \\ F_2(c,e,b) = f(d,c,e,b) - f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots f_{x_s}^{-1}(d,c,e,b) = 0 \\ F_3(c,e,b) = f(c,c,e,b) - f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots f_{y_t}^{-1}(c,c,e,b) = 0 \end{cases} \tag{8}$$

(9), (10) and (11) are systems of nonlinear equations of TSSKS in κ_n^3, κ_n^4 and κ_n^5 respectively. Modes 3-5 stand for TSSKS with one two cycles and one single cycles.

$$\begin{cases} F_1(c,e,b) = f(d,c,e,b) - f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots f_{z_r}^{-1}(e,c,e,b) = 0 \\ F_2(c,e,b) = f(e,c,e,b) - f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots f_{x_s}^{-1}(d,c,e,b) = 0 \\ F_3(c,e,b) = f(c,c,e,b) - f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots f_{y_t}^{-1}(c,c,e,b) = 0 \end{cases} \tag{9}$$

$$\begin{cases} F_1(c,e,b) = f(c,c,e,b) - f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots f_{z_r}^{-1}(e,c,e,b) = 0 \\ F_2(c,e,b) = f(e,c,e,b) - f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots f_{x_s}^{-1}(c,c,e,b) = 0 \\ F_3(c,e,b) = f(d,c,e,b) - f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots f_{y_t}^{-1}(d,c,e,b) = 0 \end{cases} \tag{10}$$

$$\begin{cases} F_1(c,e,b) = f(c,c,e,b) - f_{z_1}^{-1} \circ f_{z_2}^{-1} \circ \cdots f_{z_r}^{-1}(d,c,e,b) = 0 \\ F_2(c,e,b) = f(d,c,e,b) - f_{x_1}^{-1} \circ f_{x_2}^{-1} \circ \cdots f_{x_s}^{-1}(c,c,e,b) = 0 \\ F_3(c,e,b) = f(e,c,e,b) - f_{y_1}^{-1} \circ f_{y_2}^{-1} \circ \cdots f_{y_t}^{-1}(e,c,e,b) = 0 \end{cases} \tag{11}$$

3.2. The Parameter Calculation of TSSKS W in κ_n^m

Every TSSKS W in κ_n^m corresponds to a system of nonlinear equations which can be solved by newton-iteration method with the new word-lifting algorithm, see (6)-(11). Let $\mu = (c,e,b)^T$, $G(\mu) = (F_1(\mu), F_2(\mu), F_3(\mu))^T$, for a proper initial point $\mu_0 = (c_0, e_0, b_0)^T$, J_k in (12) is the so-called Jacobi matrix, k is the count of iteration.

$$J_k = \frac{\partial G}{\partial \mu} = \frac{\partial(F_1,F_2,F_3)}{\partial(c,e,b)} = \begin{pmatrix} \frac{\partial F_1}{\partial c} & \frac{\partial F_1}{\partial e} & \frac{\partial F_1}{\partial b} \\ \frac{\partial F_2}{\partial c} & \frac{\partial F_2}{\partial e} & \frac{\partial F_2}{\partial b} \\ \frac{\partial F_3}{\partial c} & \frac{\partial F_3}{\partial e} & \frac{\partial F_3}{\partial b} \end{pmatrix} \tag{12}$$

The nine partial derivatives in (12) can be obtained by the numerical method, for example, $\frac{\partial F_1}{\partial c} = \frac{F_1(c_k+h,e_k,b_k) - F_1(c_k-h,e_k,b_k)}{2h}$, $h = 1e-14$. According matrix form of newton-iterative method, if $|J_k| \neq 0$, then the following iteration will run forward.

$$\mu_{k+1} = \mu_k - J_k^{-1} G(\mu_k) \tag{13}$$

If $\left\| J_k^{-1} G(\mu_k) \right\|_2 < \varepsilon$, here $\varepsilon = 1e-8$, the iteration (13) will stop, parameter μ gets the fixed point. The parameters of TSSKS W in κ_n^m have been calculated successfully. However, if the iteration (13) is diverged, it can not infer that the solution does not exist, in fact, the existence and uniqueness of the solution have been confirmed by the admissibility conditions (4)-(5). So reasons only come from two aspects, one is the unfitted initial point μ_0, another is that precision is not enough to meet the requirement of the calculation process. We usually utilize the high-precision computation library, for example, the mpmath module in Python. The three parameters of six TSSKS with different modes are presented in Table 3 for example.

Table 3. Six modes of TSSKS in 1D trimodal maps

TSSKS	mode	period	c	e	b
EDC	0	3	-0.663307953130	0.698830313522	-0.624636883353
ERDLNC	0	6	-0.706764028245	0.695926583209	-0.882652903082
RDERC	1	5	-0.551472999119	0.552954797321	0.553689775309
RDRERC	1	6	-0.448194655077	0.460849583013	0.776880091485
(RE,LD,NMC)	2	7	-0.712936446668	0.686158768917	-0.856451883976
(RLE,RD,RC)	2	7	-0.395652568402	0.414763095907	0.875165513486
(LEND,RC)	3	6	-0.684459524248	0.709318815010	-0.794574082175
(MED,RLRC)	3	7	-0.643908844500	0.686173557116	0.686173557116
(EC,LD)	4	4	-0.623778012425	0.748987832140	-0.541043070225
(RERC,MD)	4	6	-0.662594849722	0.656149472629	-0.225793287451
(RDC,NME)	5	6	-0.668478449737	0.701007779708	-0.636283043328
(RLNDC,RE)	5	7	-0.685617843594	0.695821945745	-0.681951445620

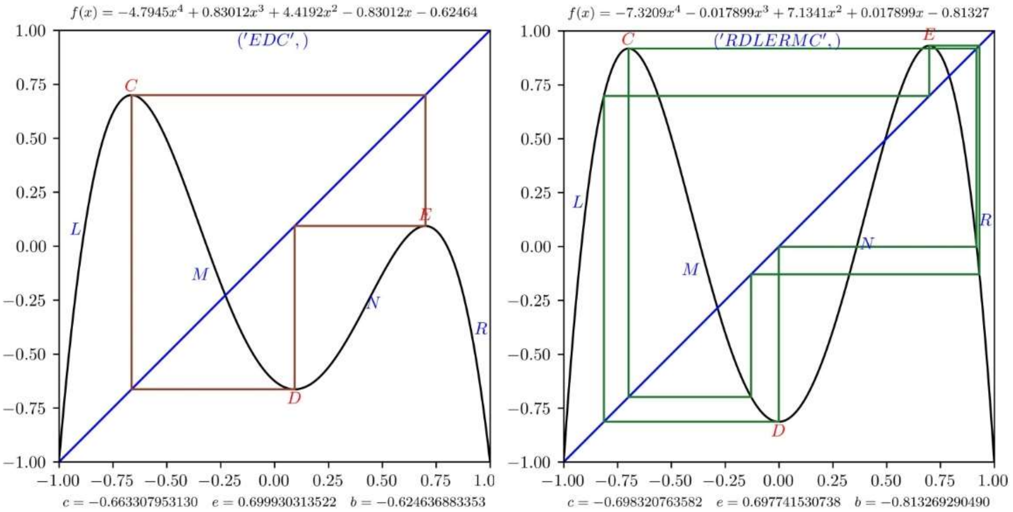

Fig. 1 Mode 0 TSSKS *EDC* (left); Mode 1 TSSKS RDLERMC (right)

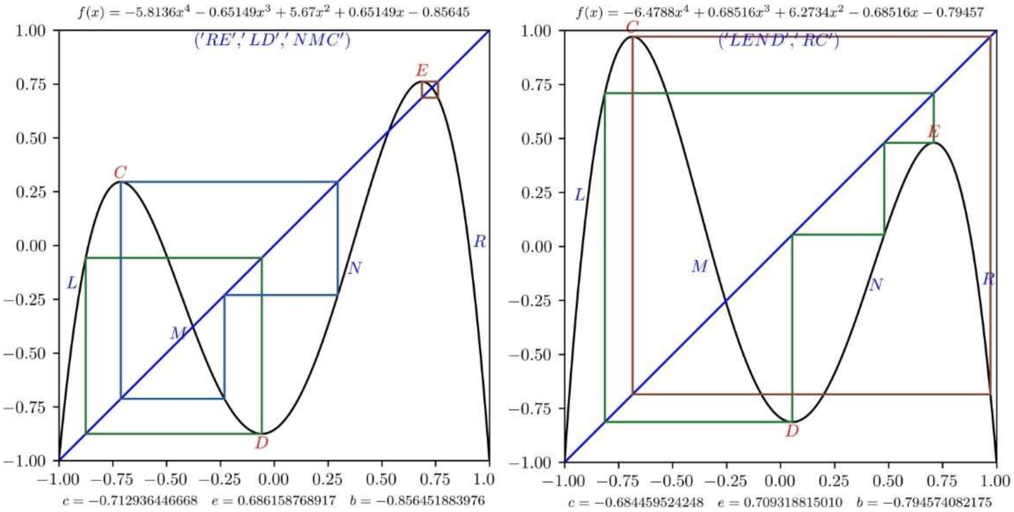

Fig. 2 Mode 2 TSSKS (RE, LD, NMC) (left); Mode 3 TSSKS (LEND, RC) (right)

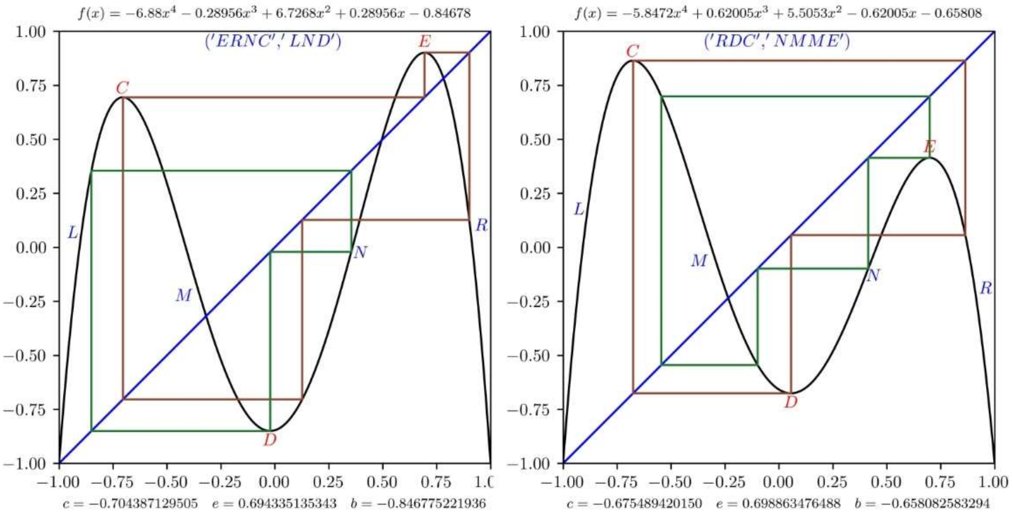

Fig. 3 Mode 4 TSSKS (ERNC, LND) (left); Mode 5 TSSKS (RDC, NMME) (right)

The result of Table 1 shows that the parameter calculation of the TSSKS with six modes is correct and trustable, Figs 1-3 have demonstrated that the solutions of (6)-(11) have attained the fixed point. However, many TSSKS in κ_n^m meet difficulties with initial sensitivity. The work on initial point should be done furtherly.

Figs 1-3 shows the numeric orbit of iterative maps corresponding six modes of TSSKS, it verified that the solutions of the nonlinear equations (6)-(11) are real and correct, it reached the numeric convergence meanwhile the symbolic convergence.

4. Conclusion

TSSKS in trimodal maps has six modes by the different patterns passing through three critical points. Mode 0 and mode 1 are important 'joints' in symbolic space and have played a key role in star products. We presented the method of producing admissible sets κ_n^m and parameter calculation method, here $n \le 13$ and $m = 0,1,2,3,4,5$. Final, we give the twelve groups of three parameters for twelve TSSKS in Table 3 and six iterative graphs numeric orbit with of TSSKS with six modes in Figures 1-3. It is worthy of noting all the results for $m \ge 2$ are presented firstly in the paper. It ensures that the symbolic space consisted of 'joints' would be generalized to the parameter space. To describe the boundaries of the parameter space, more and more TSSKS should be batch computed, however, the sensitivity to initial value will enhance the difficulty, a lot of work should be endeavored furtherly.

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